The Fourier Transform, $L^2(\mathbb{R})$, and the Riemann-Lebesgue Lemma

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The Fourier Transform on $L^1(\mathbb{R})$

$$\mathcal{F}(f)(\omega) = \hat{f}(\omega) := \int_{\mathbb{R}} f(x)e^{-2\pi i\omega x}\,dx \tag{1}$$

$$\mathcal{F}^{-1}(g)(x) = \check{g}(x) := \int_{\mathbb{R}} g(\omega)e^{2\pi i\omega x}\,d\omega \tag{2}$$

These are valid wherever the integrals are defined, for example if $f \in L^1(\mathbb{R})$

$C_o(\mathbb{R})$ and $\mathcal{S}$

Definition

$C_0(\mathbb{R})$ is the set of continuous functions with a horizontal asymptote at $f = 0$ as $|x| \to \infty$

Definition

The Schwartz Space $\mathcal{S}$ is defined as the space of C^∞ functions on $\mathbb{R}$ which, along with all of their derivatives, decay faster than any rational function.

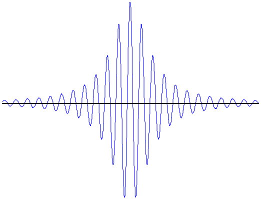
Theorems

$\mathcal{S} \overset{d}{\hookrightarrow} L^1(\mathbb{R})$; $C_0(\mathbb{R}) \subset L^\infty(\mathbb{R})$; $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ *is an isomorphism.*

The Riemann-Lebesgue Lemma

Theorem

$f \in L^1(\mathbb{R}) \Rightarrow \hat{f} \in C_0(\mathbb{R})$



The integral can be expected to be vanishingly small as the frequency increases without bound

Proof of RLL (after Nachtergaele)

Proof

By density, choose (g_n) *from* $\mathcal{S}$ *converging to* f *in* $L^1(\mathbb{R})$; *then* $(\hat{g}_n)$ *is (uniformly) Cauchy:*

$$\begin{aligned} |\hat{g}_n(\omega) - \hat{g}_m(\omega)| &= \left| \int_{\mathbb{R}} (g_n(x) - g_m(x)) e^{-2\pi i \omega x} \, dx \right| \\ &\leq \int_{\mathbb{R}} |g_n(x) - g_m(x))| \, dx \\ &= \|g_n - g_m\|_1 \end{aligned}$$

Since $\mathcal{S} \subset C_0(\mathbb{R})$ *which is complete under the sup norm, there exists a function* $h \in C_0(\mathbb{R})$ *to which* $(\hat{g}_n)$ *converges uniformly*

Proof (Cont'd)

Finally, $h = \hat{f}$ *since* $\forall\, \omega \in \mathbb{R}$,

$$\begin{aligned}
\left|h(\omega) - \hat{f}(\omega)\right| &= \lim_{n\to\infty} \left|\hat{g}_n(\omega) - \hat{f}(\omega)\right| \\
&= \lim_{n\to\infty} \left|\int_{\mathbb{R}} (g_n(x) - f(x))e^{-2\pi i\omega x}\, dx\right| \\
&\le \lim_{n\to\infty} \int_{\mathbb{R}} |g_n(x) - f(x)|\, dx = 0
\end{aligned}$$

$\square$

The Fourier Transform on $L^2(\mathbb{R})$

For $f \in L^2(\mathbb{R})$, we cannot in general define $\hat{f}$ by Equation (1).

Theorem

(Plancheral's Theorem) $f \in L^2(\mathbb{R}) \;\Rightarrow\; \|\hat{f}\|_2 = \|f\|_2$.

Now $(L^1 \cap L^2)(\mathbb{R}) \overset{d}{\hookrightarrow} L^2(\mathbb{R})$, so we can choose a sequence (f_n) in $(L^1 \cap L^2)(\mathbb{R})$ which converges to f in the 2-norm

The Fourier Transform on $L^2(\mathbb{R})$

By Plancheral's Theorem, $\|f_n - f_m\|_2 = \|\hat{f}_n - \hat{f}_m\|_2$, so $(\hat{f}_n)$ is Cauchy in $L^2(\mathbb{R})$, hence converges.

We now define $\hat{f} := \lim_{n\to\infty} \hat{f}_n$.

The inverse Fourier transform $\mathcal{F}^{-1}$ is then defined as the Hilbert space adjoint $\mathcal{F}^*$ of $\mathcal{F}$ (but not necessarily as a pointwise formula)

Fourier Inversion

First note that (by RLL) $f \in L^1(\mathbb{R}) \Rightarrow \check{f} \in C_0(\mathbb{R})$

Theorem

If $f, \hat{f} \in L^1(\mathbb{R})$, *then* $f(x) = \int_{\mathbb{R}} \hat{f}(\omega) e^{2\pi i \omega x}\, d\omega$

So if $f, \hat{f} \in L^1(\mathbb{R})$, then $f, \hat{f} \in C_0(\mathbb{R})$

Useful Sufficient Conditions for $(\mathcal{F}^{-1} \circ \mathcal{F})(f) = f$

In this case we have $f,\ \hat{f} \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$, and splitting $\mathbb{R}$ into complementary sets for which f (resp. $\hat{f}$) ≥ 1 or $f < 1$ yields that

$$f,\ \hat{f} \in L^1(\mathbb{R}) \ \Rightarrow \ f,\ \hat{f} \in \left(L^2 \cap C_0\right)(\mathbb{R}) \ \Rightarrow \ f,\ \hat{f} \in L^1(\mathbb{R})$$

So on a significant subset of $L^2(\mathbb{R})$, $\mathcal{F}^{-1} \circ \mathcal{F} = \mathcal{I}$

This set includes e^{-x^2}, $\text{sinc}^2(x)$, which are formulaic, and many others which are not

The Chernoff-Fourier Convergence Theorem

(From *Amer. Math Monthly* (1980), 399-400)

Theorem

Let f be absolutely integrable on an interval I and suppose f is Lipschitz on I with constant A. Then the (asymmetric) partial sums

$$S_{m,n}(x_0) := \sum_{k=-m}^{n} \hat{f}(k) e^{\frac{2\pi i k x}{\ell(I)}}$$

converge to $f(x_0)$ as $m, n \to +\infty$.

Proof of Pointwise Fourier Series Convergence

Proof

WLOG we can suppose that $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$, *that* $x_0 = 0$, *and that* $f(0) = 0$. *Consider the auxiliary function* $g(x) := \frac{f(x)}{e^{2\pi i x}-1}$ *and note*

$$\left|\frac{f(x)}{e^{2\pi i x}-1}\right| = \left|\frac{f(x)}{x}\frac{x}{e^{2\pi i x}-1}\right| \leq A \cdot \left|\frac{x}{e^{2\pi i x}-1}\right|$$

Also, $\left|\frac{e^{2\pi i x}-1}{x}\right| \geq 4 \;\Rightarrow\; \left|\frac{x}{e^{2\pi i x}-1}\right| \leq \frac{1}{4}$

Proof of Pointwise Fourier Series Convergence

Proof (Cont'd)

Now note that $\hat{f}(k) = \mathcal{F}(g \cdot (e^{2\pi i x} - 1))(k) = \hat{g}(k-1) - \hat{g}(k)$ *which expresses* $\hat{f}$, *so*

$$S_{m,n}(0) = \sum_{k=-m}^{n} \hat{f}(k) e^{2\pi i k(0)} = \hat{g}(-m-1) - \hat{g}(n) \tag{3}$$

which converges to $0 - 0 = 0 = f(0)$ *by the Riemann-Lebesgue Lemma applied to* g.

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